The Mathematics of Mathematics Houses (The Snaky Connection)

OMID ALI SHAHNY KARAMZADEH

This column is a forum for discussion of mathematical communities throughout the world, and through all time. Our definition of "mathematical community" is the broadest: "schools" of mathematics, circles of correspondence, mathematical societies, student organizations, extra-curricular educational activities (math camps, math museums, math clubs), and more. What we say about the communities is just as unrestricted. We welcome contributions from mathematicians of all kinds and in all places, and also from scientists, historians, anthropologists, and others.

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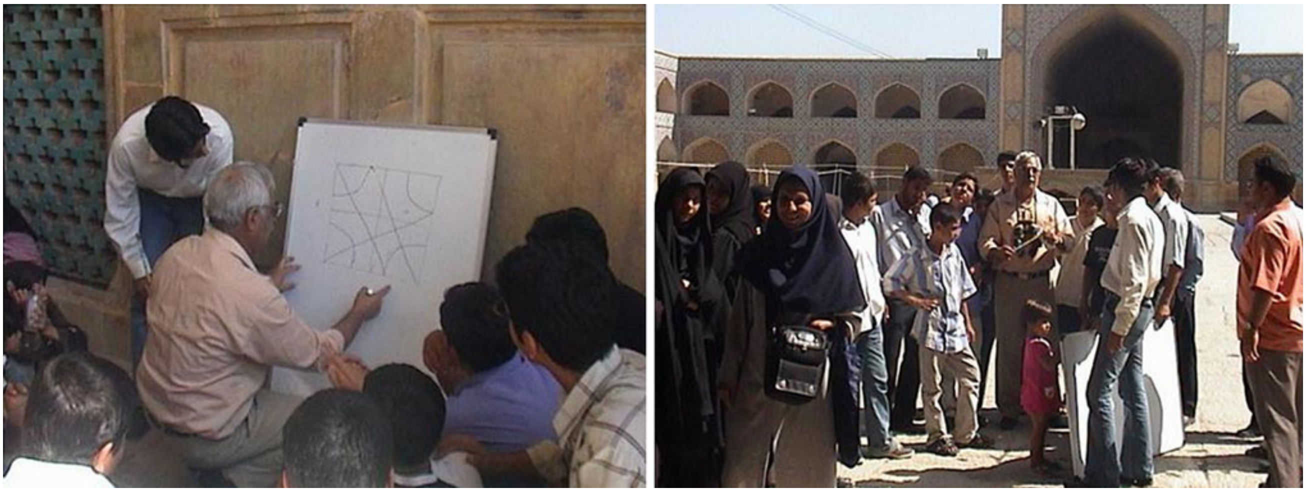
The first mathematics house in Iran was established in Isfahan in 1999 to inform the young students in this great historical and cultural city about the history, the beauty, the importance, and the applications of mathematics. It did this by familiarizing these young people with various mathematical sciences beyond the classroom, through observations, collaborations, and so forth, including teamwork with their teachers or other scholars. The house is financially supported by the municipal council of Isfahan. The activities of the Isfahan house are colorfully recounted on its lively webpage, http://www.mathhouse.org/; see also nms.lu.lv/mcg_10/2day-12/Ali_Rejali.ppt, which is the PowerPoint presentation by my colleague A. Rejali of the Isfahan University of Technology, who played a key role in establishing the Isfahan house.

In the years after the first house opened in 1999, local authorities, with the advice and the help of mathematicians, have opened houses in Tabriz, Kerman, Zanjan, and other towns in Iran. These houses are supported by the municipal councils of the cities, or by the office of the ministry of education, or by both. In a few special cases, the houses are also supported by the private sector. There is no doubt that in the long run these houses will have an impact on mathematics education in Iran and will assist the school system in raising the interest of Iranian students in mathematics. The schools and universities in different cities of Iran are usually busy with their routines and are not expected to have a serious impact on the mathematics awareness of the public, but the mathematics houses, in collaboration with the Iranian Mathematical Society, can have a natural and key role in increasing the public understanding of and appreciation for mathematics. (See [4], [1] for more details about the mathematics houses in Iran.)

One of the main activities pursued by those in the Isfahan Mathematics House (IMH), which is being emulated in other mathematics houses in Iran, is for the leaders to invite prominent scholars from both inside the country and from abroad to deliver popular talks about mathematics to general audiences, including mathematicians, teachers, students, and laypeople. This provides a good opportunity for the young students to meet some well-known mathematicians, which is very exciting for the students. Past invitees from abroad include Y. Dodge (University of Neuchâtel, Switzerland), who spoke on π and random-number generators; S. Gazor (Queen's University, Canada) on psycho-acoustics; K. Salamatian (Paris VI, France) on internet measurement; A. Enayat (American University, USA) on a gift from Cantor's heaven; J. Vaananen (Helsinki University, Finland) on logic and mathematics; M. Andler (Université de Versailles-Saint-Quentin and Animath, France) on current problems in mathematics education in France; G. Frey (Duisburg-Essen University, Germany) on number theory and coding; J.-M. Deshouillors (Université de Bordeaux 1, France) on prime numbers yesterday and today

DOI 10.1007/s00283-012-9337-5



(Left) Students working with their teacher at the Isfahan Mathematics House. (Right) Students working with their teacher at the site of a very old building in Isfahan (in the workshop of mathematics and arts, organized by members of the Isfahan Mathematics House, 2008) to learn about the role of geometry in the architecture of these ancient buildings.

(2009); T. Goris (Utrecht University, the Netherlands) on education in the Netherlands; A. Sabery (Stanford University, USA) on the internet mathematical model; L. Berggren (Simon Fraser University, Canada) on the lost works of the mathematician Abu Sahl Koohi; J. Mason (Open University, England) on teaching mathematics as a constructive and creative activity; K. Stacy (Melbourne University, Australia) on teaching mathematical thinking to students.

In my opinion, the mathematics described in these popular presentations, whether recreational, educational, or for the advancement of mathematics, should generally be about mathematics, not necessarily describing the details of mathematics itself (see the "Aims and Scope" of *The Mathematical Intelligencer* for the meaning of "about mathematics"). See also [10], [16].

The main aim of these talks, like the policy of these houses, must be to attract people and to make these houses a home for people interested in knowing more about mathematics – and also a place for those who don't want to know about mathematics because they are afraid of it. We want to turn these houses into stomping grounds for those "fearful" young people, too. There must always be some activities in these houses that they can enjoy. We should remind ourselves that:

(1) Most people who enjoy the art of painting cannot paint.
(2) Most people who enjoy music (whether classical or pop) are neither singers nor can they play any instruments.
(3) Many people who enjoy watching sports, plays in theaters, magic shows, or other events, have no previous knowledge or experience in any of these activities.

In particular, in these houses we should not talk about the citation index, impact factors, the numbers of articles one has written, the professorships, the readerships, or one's degrees. Fortunately, most of the people who enjoy talking about these things are not the regular patrons of these houses, and they have no interest in the activities of the houses. Because most young people at the present time are studying English, the libraries of these houses should subscribe to mathematics journals such as *The Mathematical Intelligencer*, *Plus Magazine*, *American Mathematical Monthly*, *Mathematics Magazine*, *The College Mathematics Journal*, *The Mathematical Gazette*, *Crux Mathematicorum*, or others.

The first mathematics house in Tehran was established in 2009, and it was financially supported by the Ministry of

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OMID ALI SHAHNY KARAMZADEH graduated from Tehran university in 1969 and was granted a scholarship to study abroad. Since receiving his Ph.D. in 1974 (with D. Rees at the University of Exeter, United Kingdom), he has been teaching at Shahid Chamran University of Ahvaz (formerly Jondishapour University). He works in ring theory, topology, and mathematics education, Pólya style. His enthusiasm for mathematics (especially, for geometry, number theory, and problem-solving) dates back to his early school days. He has been involved in the Iranian Mathematical Olympiad and Iranian University Students Mathematics Competitions for several years. His popular talks in mathematics (in Persian) are collected in two volumes, and he has been awarded many prizes. His hobbies include jogging with friends regularly and playing poker with them occasionally (for fun, not money).

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(Left) Professor Peter Taylor (University of Canberra, Australia) during a problem-solving workshop, IMH (2007). (Right) Professor Abbas Edalat (Imperial College, England), during a popular presentation at IMH (2004).

The building containing the Tehran Mathematics House.

Education; its activities are currently limited. A second house, supported by the Municipal Council of Tehran, opened in 2012. We may need still more mathematics houses in the future in this big city. The population of Iran is still rather young, and the number of students and young people living in Tehran, with respect to the rest of the country, is very high.

I was invited to deliver a popular talk at the inauguration of Tehran's second mathematics house. Because these houses welcome new ideas, I used the occasion to make some suggestions. I will discuss four of them.

First, we must address the notorious fear of mathematics among some young people and find a way to help them understand that mathematics itself is not something to be feared at all, although the exams can induce fear. People should know that most mathematicians cannot deal seriously with the mathematics outside their specialties; if each of us had to take exams in areas outside our specialties, we would be fearful and anxious, too.

Most high-school students, and most of their teachers, think that a first-rate mathematician is someone who can find the square root of any positive real number, factor any large integer number in a second, and solve any problem in geometry, arithmetic, or in any part of mathematics off the top of his or her head. They believe that mathematicians are born, not made, and that they themselves will never be good at mathematics. Students usually develop such mistaken ideas, and their fear of mathematics, in school. Incompetent or poorly trained teachers can pass their fear and lack of understanding on to their students. Thus a goal in these mathematics houses should also be to help such teachers increase the mathematical knowledge needed for their jobs.

Second, in contrast to other disciplines of the basic sciences, there are many cranks in the mathematics field (i.e., eccentric people who think they have made some important discovery in mathematics, such as trisecting angles, proving Fermat's Last Theorem in a few lines, or duplicating the cube). Before the existence of mathematics houses, these people used to write to different mathematics departments about their discoveries and even come to our Annual Iranian Mathematics Conference (AIMC) to talk to people about their inventions. But now it is natural for some of them to show up at these houses looking for someone to talk to. People at these houses should deal with them very carefully, rightly, and politely. We should not encourage their cranky ideas and beliefs; but when we do not understand them, we should admit it and let the others, who can understand them, show them their mistakes. Indeed it is helpful for teachers and students to understand the flaws in their proofs. I can recall a crank who was a regular participant in our AIMC for two decades, until he passed away; some of my colleagues and I used to enjoy talking to him during those conferences. These houses should keep lists of professionals and volunteers who can be called on to deal with the mildly eccentric (see [9]).

Third, operators of mathematics houses should run the useful "Tournament of Towns" (TT), prevailing on schools in their towns and in the neighboring suburbs to encourage their students to participate in this international competition. This competition can have more impact on the education of mathematics in Iran than the IMO. In contrast to the IMO, the TT may provide the opportunity to many students from remote parts of the country to compete with students from around the world, and at the same time to experience a good cultural exchange without traveling abroad.

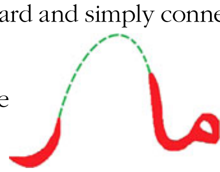
The TT, now an international mathematics competition, started in the Soviet Union in 1979. At first it was an informal small event, a competition for the students of big cities such as Moscow, Leningrad, Kiev, and smaller towns that did not participate in the All-Union Mathematical Olympiad, the popular USSR mathematics competition of that time. Yambol, Ruse, Varana, Sofia, and other Bulgarian cities joined the TT in 1984; Canberra, Australia, joined the TT in 1988, and other Australian cities such as Melbourne, Newcastle, and Hobart now participate in this competition. With the active participation of Eastern European countries and the recent interest shown by cities in the west and in southeast Asia, the TT is truly an international mathematics competition. Although it differs from the IMO, it is becoming equally important. (For more details about the history, the rules, and the philosophies of this Tournament, see [13]).

Fourth (last but not least), as we all know, in the latter half of the 20th century (thanks to Bourbaki), geometry fell out of favor in many countries and in particular in Iran. The mathematics houses should promote geometry gradually, and they should help bring it back to the schools. To achieve these goals, we should first help some teachers become qualified to teach geometry, by holding workshops and seminars, and by presenting mini-courses for them; these are first steps toward closing the existing gaps between university and high-school mathematics education.

A main aim of these houses should also be to convey mathematical thinking and concepts, in an easy way, to others, and to attract them to become curious about mathematics. An Iranian anecdote is appropriate here.

The story of مار and  (note, snake is called Mar and written مار, in Farsi)

Two individuals were asked to write "snake" (i.e., مار) before an audience. One of them wrote the word correctly مار. The other, who was illiterate, drew the figure of a snake instead. Most of the audience thought the latter was fine.

People usually interpret this story to mean that the illiterate person gained that approval by deception. But I disagree. Their approval shows that most of the audience was illiterate, too. Also, in the audience, there might have been some who had never seen snakes before; the drawing introduced this dangerous reptile to them (although unintentionally and even with the aim of trickery). But our educated person, who had written correctly مار, could go one better: after seeing this apparent chicanery, he could go back to his own writing مار on the blackboard and simply connect the two parts ر and ما to get the shape  of a snake. He could say that the writing of Mar is مار and  is the shape of it.

Let us call the latter connection the *snaky connection*. This is what we should do in mathematics, that is, make snaky connections. That is why triangles should have a very special place in these mathematics houses. There is no better place to look for snaky connections than in the vast lore of triangles.

No object has ever served mathematics better or longer. Compare the number of nontrivial results that are true for all topological spaces, rings, groups, etc., without putting extra assumptions on them with the number of nontrivial results that are true in any triangle. I know of no nontrivial result that holds in every topological space[1] As for rings, one might mention that each ring with an identity has maximal ideals, but we have to be careful here, for this is exactly the Axiom of Choice. I leave it to you to find universal nontrivial results about other mathematical objects. When it comes to deducing results in mathematics just from the definition of an object, nothing can hold a candle to the triangle. The triangle will serve mathematics forever.

Obtaining new results in triangles is as easy as swiping your card through the slot of an ATM to get money. But of course, just as you must have something in the bank to be able to use your card, you must also have some "investment" in Euclidean geometry to be able to work with the triangle. This investment should start early in school. For even if you are a Fields medalist, if you did not learn in your school days that the angle between the angle bisector of the angle $\angle A$ and the altitude through the vertex A in a triangle $\triangle ABC$ is equal to $\left|\frac{\angle B}{2} - \frac{\angle C}{2}\right|$, and hence the angle between the median and the altitude through the vertex A is not less than $\left|\frac{\angle B}{2} - \frac{\angle C}{2}\right|$, it will take you time to guess this very trivial fact. Even more, it will take time to guess the properties of Fermat's point, the symmedian (or Lemoine) point, the nine-point circle, pedal triangle, orthic triangle, and thousands of other nontrivial and interesting facts about triangles, when you are over 40. Think of sports: if you did not play soccer when you were young, you can't start in middle age; you might seriously hurt your knees.

So let's talk about triangles and snaky connections.

We all know that the three positive real numbers $a \leq b \leq c$ are the sides of a triangle if and only if $c < a + b$. Now we may ask, can we have a system of linear equations involving three positive real numbers a, b, c whose solvability in real numbers, is equivalent to the existence of a triangle whose sides are these three numbers? The following simple fact, whose proof is too trivial to be given, settles the question.

THEOREM 1 *If a, b, c are real numbers, then the system*

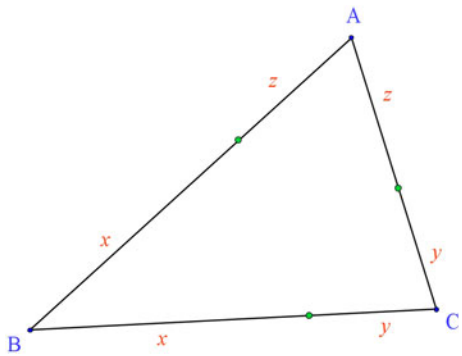
$$\begin{cases} a = x + y \\ b = y + z \\ c = z + x \end{cases}$$

has positive solutions for x, y, z if and only if a, b, c are the sides of a triangle.

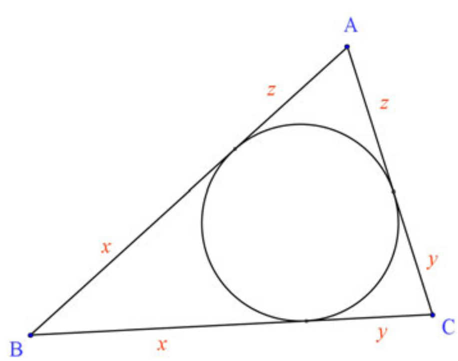
These equations are like writing "Mar" three times, and finding x, y, z is also like writing some parts (but we do not know which parts) of "Mar" (algebraic manipulation). But

[1] If the unsettled problem of finding the number of all the possible topologies that can be defined on a finite set is resolved, it will be a combinatorial fact, not a topological result, and still not a fact but all topological spaces.

where is the snake – the triangle – itself? In other words can we ask that person who is seemingly presenting the fraud to depict this snake for us, but this time with these three parts on its body? (surely no illiterate person can do this depiction, even if he or she is a clever charlatan). Consequently, we infer that the algebraic solution (i.e., writing just Mar) sometimes does not reveal everything per se. One can always ask about the geometric interpretation of the previously mentioned equations. This motivates us to try to draw the shape of this snake (i.e., the triangle).



This algebraic solution tells us that x, y, and z, the separating points on the sides of the triangle, are unique. But if we stop with this algebraic proof, leaving unanswered the latter question (what are x, y, z geometrically?), then we are giving in to Bourbaki ("Down with Euclid! Death to triangles!"); see [19]. Here is the snake: x, y , z are nothing but the segments that are produced by the incircle on the sides of the triangle! In fact, the following shape of the snake (i.e., the triangle) is the complete geometric proof of the previous theorem, which also settles the latter question and the uniqueness of the above points.



This is the genuine "snaky connection", which is done frequently in algebraic geometry. Thus writing "Mar" means algebra, and drawing the snake means geometry, and "snaky connection" means algebraic geometry. Generally, pegging a particular algebraic property of the algebra of polynomials, in a finite number of variables over a field, to a geometric property in the related affine space over the latter field, has always been of main interest to the workers in the area of algebraic geometry. Perhaps the starting point here is the snaky connection, which says that the ideal generated by any set of polynomials in the aforementioned algebra of polynomials is generated by a finite number of these polynomials (this is called Hilbert's Basis Theorem). Although it seems purely an algebraic result, it is in fact a beautiful, snaky connection, which is a basic result and one of the fundamental cornerstones in algebraic geometry. As a consequence of this fundamental snaky connection, defining an algebraic hypersurface to be the zero-set of a real polynomial in n real variables, one can immediately infer that for any set of such polynomials the intersection of the surfaces they thus define is another one (Bear in mind that every finite set is trivially a hypersurface, and when $n = 2$ the terminology "algebraic curve" is geometrically more appropriate, and is customary). Is there anybody with just the knowledge of school mathematics who might not be fascinated by the beauty of the latter fact?

Now, let us return to our simple theorem. We note that there are triangles for which the sum of two sides can be arbitrarily larger than (respectively, arbitrarily closer to) the third one. To see this, consider a very tall isosceles triangle (respectively, an isosceles triangle with an angle as near to 180° as we need). We can have a large class of examples, not necessarily isosceles triangles, of the latter kind: for any triangle $\triangle ABC$, with $\angle A \geq 90°$, we always have $b + c < a + h_a$, where h_a is the altitude through the vertex A, which can be as small as possible by taking $\angle A$ to be as close to 180° as possible. Consequently, if a, b, c are the sides of a triangle, and we put

$$\begin{cases} \frac{a+b}{c} = 1 + \alpha \\ \frac{a+c}{b} = 1 + \beta \\ \frac{b+c}{a} = 1 + \gamma \end{cases}$$

then in some triangles some of α, β, γ can be arbitrary large (resp., small). But in general, we remember the following interesting and well-known fact (but not, of course, in the form that follows) for all triangles, which is an easy consequence of our Theorem 1:

THEOREM 2 *Let α, β, γ be the aforementioned values. Then $\alpha\beta\gamma \leq 1$.*

PROOF. We are to show that

$$\left(\frac{a+b}{c} - 1\right)\left(\frac{a+c}{b} - 1\right)\left(\frac{b+c}{a} - 1\right) \leq 1$$

or

$$(a + b - c)(a + c - b)(b + c - a) \leq abc$$

– this is known as the Lehmus inequality [8] and Padoa's inequality, see [18] and [17, Appendix I]. Now if we go back to our Theorem 1, and transfer its equations with their solutions, which are $x = \frac{a+c-b}{2}$, $y = \frac{a+b-c}{2}$, $z = \frac{b+c-a}{2}$, to the latter inequality, we must show that

$$8xyz \leq (x + y)(y + z)(x + z).$$

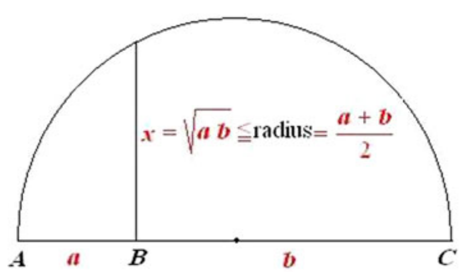
And this is evident by the AM-GM inequality ([2], [11], [15], and [14]).

This theorem is equivalent to the AM-GM inequality for two variables. To see this, let us suppose that $a \leq b$ are positive real numbers, hence there is a triangle with the sides a, b, b. Taking $c = b$, we have $a^2(2b - a) \leq ab^2$, which immediately implies the AM-GM inequality for two variables. The proof of the aforementioned theorem yields the converse.

As another example of the richness of triangles, let us prove that of all triangles with a given area, the equilateral has the smallest perimeter; see [17], p. 6, [14], [15]. Let S be the area of a triangle $\triangle ABC$ with the sides a, b, c, then by Heron's formula we have $S^2 = P(P-a)(P-b)(P-c)$, where P is the semiperimeter (i.e., $P = \frac{a+b+c}{2}$). Although there are many different proofs in the literature of Heron's formula, let us digress for a moment, and provide a quick one, too. We have $4S^2 = b^2c^2\sin^2 A = b^2c^2(1+\cos A)(1-\cos A)$; replace $\cos A$ by $\frac{b^2+c^2-a^2}{2bc}$, and we are done. Now if S is fixed, then $2P$ is minimum if the sum of the four variables P, $P-a$, $P-b$, $P-c$, which is $2P$, is minimum, and this occurs if $P = P-a = P-b = P-c$, that is, $a = b = c = 0$, which is absurd. But one can easily get around this stumbling block. We have $\frac{1}{3}S^2 = \frac{1}{3}P(P-a)(P-b)(P-c)$. Consequently, $2P$ is minimum if $\frac{1}{3}P + (P-a) + (P-b) + (P-c) = \frac{4}{3}P$ is minimum, and this occurs if $\frac{1}{3}P = P-a = P-b = P-c$, hence $a = b = c = \frac{2}{3}P$, and we are done (see also [12, p. 99, Problem 11]).

Don't you think Heron's formula, together with the AM-GM inequality, acts as a two-headed snaky connection in problems of this kind? Is $\frac{1}{3}$ a Deus ex Machina? Or is it inevitable? See [5, Inevitability and Unexpectedness].

In fact, the AM-GM inequality for two variables is itself a "snaky connection", as we see in the following well-known snakelet.



I conclude with a word of warning. Although genuine snaky connections can be extremely useful in mathematics, watch out for artificial ones, such as the depiction of a strange creature that has the body of a snake, the head of a lion, the eyes of an owl, and the braying of a donkey. We should remember that we all gather in these kinds of places to try to make mathematics more accessible to others and to make it look as easy as possible, not to create horrible objects that frighten people. We must not let the mathematics houses become Snaky Horror Houses. Let me provide a mathematical prototype of the latter strange creature, which should not be allowed for presentation by anyone in the mathematics houses (except, of course, by me, and just this time!). Take the triangle $\triangle ABC$ with the sides $\sqrt[2]{2}$, $\sqrt[3]{3}$, and $\sqrt[5]{5}$. We can easily write the third-degree equations $f_i(x) = 0, i = 1, 2, \ldots, 9$, whose roots are the three sides, the three altitudes, the three segments produced by the incircle on the sides, the three exradii, the sines of the angles, the cosines of the angles, the squares of the sides, the lengths of the medians, the lengths of the angle bisectors of the triangle $\triangle ABC$, respectively; see [3], [6], [7]. Now put $g = \sum_{i=1}^{9}(f_i(x_i))^8$ and write it as a single polynomial, and also put $h = f_1(x_1)\cdot f_2(x_2)\cdot f_3(x_3)\cdot f_4(x_4)\cdot f_5(x_5)\cdot f_6(x_6)\cdot f_7(x_7)\cdot f_8(x_8)\cdot f(x_9)$ and multiply out these polynomials to obtain h as a single polynomial too. We now have the following strange creature.

Theorem 3 *Let h, g be the previously mentioned polynomials in 9 variables. Then there are infinitely many triangles (resp., a unique triangle) such that if we replace $x_1, x_2, \ldots x_9$ by any side, any altitude, any of the segments produced on the sides by the incircle, any of the exradii, sine of any angle, cosine of any angle, the square of any side, the length of any median, the length of any angle bisector of these triangles (resp., of this unique triangle) in h (resp., in g), respectively, we obtain $h = 0$ (resp., $g = 0$).*

ACKNOWLEDGMENTS

I thank the referee for reading the article carefully and for providing useful suggestions for improvements. Thanks are also due to Professor M. Senechal for dealing with different revised versions of the article critically and patiently. I would also like to thank my colleague, Dr. M. Namdari, and my Ph.D. student, O. Ghayour, for their TEXnical support in preparing the TEX file of this article. Last, but not least, I thank my colleague, Professor A. Rejali, for bringing [1] to my attention and helping me provide some pictures related to the mathematics houses in Isfahan and Tehran.

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