Representation Theory of Finite Groups and Its Problems

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Abstract

Let $\mathbb{F}$ be a field. We have two important examples of $\mathbb{F}$-algebras: $\mathrm{Mat}(n, \mathbb{F})$, the $\mathbb{F}$-algebra of $n \times n$ matrices with entries in $\mathbb{F}$ and, for every finite group G, the **group algebra**

$$\mathbb{F}G = \{\sum_{g \in G} a_g g \mid a_g \in \mathbb{F}\}$$

with the multiplication of G extended linearly to $\mathbb{F}G$. The **representation theory of finite groups** studies the homomorphisms between $\mathbb{F}G$ and $\mathrm{Mat}(n, \mathbb{F})$. In this talk, we state some main problems of the representations of finite groups.

1 Introduction

Representation theory of finite groups provides a powerful tool for proving theorems about finite groups. In fact, there are some important results, such as "**Frobenius' theorem**", for which no proof without representations is known.

Theorem 1. *(Frobenius) Let G be a Frobenius group with complement H. Then there exists $N \trianglelefteq G$ with $HN = G$ and $H \cap N = 1$.*

Let $\mathbb{F}$ be a field. We have two important examples of $\mathbb{F}$-algebras: $\mathrm{Mat}(n, \mathbb{F})$, the $\mathbb{F}$-algebra of $n \times n$ matrices with entries in $\mathbb{F}$ and, for every finite group G, the **group algebra**

$$\mathbb{F}G = \{\sum_{g \in G} a_g g \mid a_g \in \mathbb{F}\}$$

with the multiplication of G extended linearly to $\mathbb{F}G$. An $\mathbb{F}$-representation of the group G is an algebra homomorphism $\rho : \mathbb{F}G \to \mathrm{Mat}(n, \mathbb{F})$. The degree of the representation ρ is n. Notice that if A and B are $\mathbb{F}$-algebras, then an algebra homomorphism is an $\mathbb{F}$-linear, multiplicative map $f : A \to B$ such that $f(1_A) = 1_B$.

If $V = \mathbb{F}^n$ and $\rho : \mathbb{F}G \to \mathrm{Mat}(n, \mathbb{F})$ is an $\mathbb{F}$-representation of G of degree n, then we may define $vg = v\rho(g)$ for $v \in V$ and $g \in G$, and V becomes an $\mathbb{F}G$-module. Conversely, suppose that V is an $\mathbb{F}G$-module and let $\{v_1, \ldots, v_n\}$ be a basis of V. If for every $g \in G$, we write

$$v_i g = \sum_{j=1}^{n} a_{ij} v_j,$$

then the map $\rho(g) = (a_{ij})$ is an $\mathbb{F}$-representation of the group G. Because of this correspondence, the study of $\mathbb{F}G$-modules is equivalent to the study of $\mathbb{F}$-representations of the group G.

It is easy to check that two $\mathbb{F}G$-modules V_1 and V_2 are isomorphic if and only if any two associated representations ρ_1 and ρ_2 are similar; which means that there exists an invertible matrix $P \in \mathrm{GL}(n, \mathbb{F})$ such that

$$\rho_1(g)P = P\rho_2(g)$$

for all $g \in G$. Hence, an $\mathbb{F}$-representation of G uniquely determines an $\mathbb{F}G$-module up to isomorphism and an $\mathbb{F}G$-module uniquely determines an $\mathbb{F}$-representation of G up to similarity.

Also, an $\mathbb{F}$-representation ρ of G is **irreducible** if its corresponding module is a simple module.

Let A be an $\mathbb{F}$-algebra. Then A itself is an A-module under right multiplication. This module is called the regular A-module and is denoted by A°. An A-module V is a completely reducible module if for every submodule $W \leqslant V$, there exists another submodule $U \leqslant V$ such that $V = W \oplus U$. We will assume that every A-module has finite dimension over $\mathbb{F}$. An $\mathbb{F}$-algebra A is semisimple if its regular module, A° is completely reducible. Equivalently, an $\mathbb{F}$-algebra A is semisimple if $\mathrm{J}(A) = 0$, where $\mathrm{J}(A)$ is the Jacobson radical of A, the intersection of all maximal right ideals of A. Also, we say that an $\mathbb{F}$-algebra A is simple if it has no proper (two sided) ideals.

Theorem 2. *(Maschke) Let G be a finite group and $\mathbb{F}$ a field whose characteristic does not divide $|G|$. Then the group algebra $\mathbb{F}G$ is semisimple and every $\mathbb{F}G$-module is completely reducible.*

The Wedderburn's theorem classifies the semisimple algebras over algebraically closed fields: they are direct sums of matrix algebras.

Theorem 3. *(Wedderburn) Suppose that A is a semisimple algebra over $\mathbb{F}$, an algebraically closed field. Then*

1. *There is only a finite set $\{B_1, \ldots, B_n\}$ of distinct minimal ideals of A. Also,*

$$A = \bigoplus_{j=1}^{n} B_j.$$

2. *If M_j is a minimal right ideal of A contained in B_j, then $\{M_1, \ldots, M_n\}$ is a complete set of representatives of pairwise nonisomorphic minimal right ideals of A. In particular, $\{M_1, \ldots, M_n\}$ is a complete set of representatives of pairwise nonisomorphic simple A-modules. Moreover,*

$$ann(M_i) = \sum_{j \neq i} B_j.$$

3. $\dim_{\mathbb{F}}(\mathrm{Z}(A)) = n$.

4. *The natural map induced by right multiplication yields an isomorphism $B_j \cong \mathrm{End}_{\mathbb{F}}(M_j)$. In particular,*

$$\dim_{\mathbb{F}}(A) = \sum_{j=1}^{n} (\dim_{\mathbb{F}}(M_j))^2.$$

5. *The ideal B_j is the direct sum of $\dim_{\mathbb{F}}(M_j)$ minimal right ideals of A isomorphic to M_j.*

Let $\rho : \mathbb{F}G \to \mathrm{Mat}(n, \mathbb{F})$ be an $\mathbb{F}$-representation. Then the $\mathbb{F}$-character χ of G afforded by ρ is the function given by $\chi(g) = \mathrm{trace}(\rho(g))$. If $\mathrm{char}(\mathbb{F}) = 0$, then the degree of the character χ is $\chi(1) = n \cdot 1_{\mathbb{F}} = n$ as $\rho(1) = I_n$.

$\mathbb{F}$-characters of degree 1 are called linear characters. In particular, the function 1_G with constant value 1 on G is a linear $\mathbb{F}$-character. It is called the principal $\mathbb{F}$-character.

Similar $\mathbb{F}$-representations of G afforded equal characters and characters are constant on the conjugacy classes of a group.

Let G be a finite group and $\mathbb{F}$ be an algebraically closed field with $\mathrm{char}(\mathbb{F}) = 0$. By Maschke and Wedderburn's theorems, there is only a finite set $\{B_1, \ldots, B_n\}$ of distinct minimal ideals of $\mathbb{F}G$ and if M_j is a minimal right ideal of $\mathbb{F}G$ contained in B_j, then $\{M_1, \ldots, M_n\}$ is a complete set of representatives of pairwise nonisomorphic simple $\mathbb{F}G$-modules. Assume that ρ_j is an associated $\mathbb{F}$-representation of M_j. Then $\{\rho_1, \ldots, \rho_n\}$ is a complete set of representatives of pairwise nonsimilar irreducible $\mathbb{F}$-representations of G. Let χ_j be the $\mathbb{F}$-character afforded by ρ_j. Then the set

$$\mathrm{Irr}_{\mathbb{F}}(G) = \{\chi_1, \ldots, \chi_n\}$$

is the set of all irreducible of $\mathbb{F}$-characters of G (that is, $\mathbb{F}$-characters afforded by irreducible $\mathbb{F}$-representations). It can be proved two $\mathbb{F}$-representations of the group G are similar if and only if they afford equal $\mathbb{F}$-characters.

Since $\dim_{\mathbb{F}}(\mathbb{F}G) = |G|$, $\dim_{\mathbb{F}}(M_j) = \deg(\rho_j) = \chi_j(1)$, and

$$\dim_{\mathbb{F}}(\mathbb{F}G) = \sum_{j=1}^{n} (\dim_{\mathbb{F}}(M_j))^2,$$

we have

$$|G| = \sum_{j=1}^{n} \chi_j(1)^2.$$

It seems natural at this point to ask how we can determine the integer n purely group theoretically, without looking at representations. By Wedderburn's theorem we know that $n = \dim_{\mathbb{F}}(\mathrm{Z}(\mathbb{F}G))$.

Theorem 4. *Let $K_1, K_2, \ldots, K_n$ be the conjugacy classes of a group G. Let $S_j = \sum_{x \in K_j} x \in \mathbb{F}G$. Then the set $\{S_1, \ldots, S_n\}$ forms a basis for $\mathrm{Z}(\mathbb{F}G)$. In particular, $|\mathrm{Irr}_{\mathbb{F}}(G)| = k(G)$, where $k(G)$ is the number of conjugacy classes f G.*

Corollary 5. *Let G be a finite group and $\mathbb{F}$ an algebraically closed field with $\mathrm{char}(\mathbb{F}) = 0$. Suppose that $\mathrm{Irr}_{\mathbb{F}}(G) = \{\chi_1, \ldots, \chi_{k(G)}\}$ is the set of all irreducible of $\mathbb{F}$-characters of G. Also, assume that for all $1 \leqslant j \leqslant k(G)$, $n_j = \chi_j(1)$. Then*

$$\mathbb{F}G \cong \bigoplus_{j=1}^{k(G)} \mathrm{Mat}(n_j, \mathbb{F}).$$

Suppose that $\mathbb{F}$ is an algebraically closed field with $\mathrm{char}(\mathbb{F}) = 0$. Let $\mathrm{Irr}_{\mathbb{F}}(G) = \{\chi_1, \ldots, \chi_{k(G)}\}$ be the set of all irreducible of $\mathbb{F}$-characters of G with $n_j = \chi_j(1)$. The **character degree set** of group G is the set

$$\mathrm{cd}(G) = \{n_j; 1 \leqslant j \leqslant k(G)\} = \{\chi(1); \chi \in \mathrm{Irr}_{\mathbb{F}}(G)\}.$$

It is well known that the character degree set $\mathrm{cd}(G)$ can be used to obtain information about the structure of the group G. In this field of study, there are two main general problems that arise:

Problem 1. Which sets of positive integers containing 1 can occur as $\mathrm{cd}(G)$ for some finite group G?

Problem 2. If there is some set of positive integers X containing 1 so that $X = \mathrm{cd}(G)$, then what can be said about the structure of G?

2 Known Results

Let $\mathbb{F}$ be an algebraically closed field with $\mathrm{char}(\mathbb{F}) = 0$. In general, determining the character degrees of a finite group over $\mathbb{F}$ is not an easy task but there are some tools that help to determine some of them.

- $\sum_{\chi \in \mathrm{Irr}_{\mathbb{F}}(G)} \chi(1)^2 = |G|$.
- For all $\chi \in \mathrm{Irr}_{\mathbb{F}}(G)$, $\chi(1)$ is a divisor of $|G : Z(G)|$. In fact, if K is normal abelian subgroup of G, then $\chi(1)$ is a divisor of $|G : K|$.
- $\mathrm{cd}(S_3) = \{1, 2\}$, $\mathrm{cd}(Q_8) = \{1, 2\}$, and $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$.
- If N is a normal subgroup of G such that G/N is a Frobenius group with abelian kernel, then $|G : N| \in \mathrm{cd}(G)$. In fact, in this case, $\mathrm{cd}(G/N) = \{1, |G : N|\}$. Hence, for all positive integers m, there exist a group G with $\mathrm{cd}(G) = \{1, m\}$.
- If p is any prime number, then there is a group G such that

  $$\mathrm{cd}(G) = \{1, p^{n_1}, \ldots, p^{n_k}\},$$

  where $n_1, \ldots, n_k$ are arbitrary positive integers greater than 2.
- If $G = A \times B$, then

  $$\mathrm{cd}(G) = \{mn; m \in \mathrm{cd}(A) \text{ and } n \in \mathrm{cd}(B)\}.$$

- For any two relatively prime integers $m, n \geqslant 2$, there is a group G which is not direct product of two nonabelian subgroups and $\mathrm{cd}(G) = \{1, m, n, mn\}$.
- (Aziziheris- 2015) Let a, b, c, and d be pairwise relatively prime integers greater than 1. Then the equation $\mathrm{cd}(G) = \{1, a, bd, cd\}$ has no solution in finite groups.

In general, the structure of the character degree set of G does not completely determine the structure of G, but it gives us some information about the structure of G. For examples:

- $\mathrm{cd}(D_8) = \mathrm{cd}(Q_8) = \{1, 2\}$.
- A group G is abelian if and only if $\mathrm{cd}(G) = \{1\}$.
- A group G has a normal abelian p-complement if and only if $\mathrm{cd}(G) = \{1, p^{n_1}, \ldots, p^{n_k}\}$, where $n_1, \ldots, n_k$ are arbitrary positive integers.
- **Thompson's theorem** implies that if a prime number p divides $\chi(1)$ for every non-linear $\chi \in \mathrm{Irr}_{\mathbb{F}}(G)$, then G has a normal p–complement.
- By **Itô-Michler's theorem**, a group G has a normal abelian Sylow p–subgroup if and only if every element of $\mathrm{cd}(G)$ is relatively prime to p.
- (Lewis- 1998) Let p, q, and r be distinct primes. If G is a finite group with $\mathrm{cd}(G) = \{1, p, q, r, pq, pr\}$, then the group $G = A \times B$ is the direct product of two normal nonabelian subgroups A and B of G such that $\mathrm{cd}(A) = \{1, p\}$ and $\mathrm{cd}(B) = \{1, q, r\}$.
- (Aziziheris- 2010) Let q and r be distinct primes and $m > 1$ an integer not divisible by q or r. If G is a finite group with $\mathrm{cd}(G) = \{1, m, q, r, mq, mr\}$, then the group $G = A \times B$ is the direct product of two normal nonabelian subgroups A and B of G such that $\mathrm{cd}(A) = \{1, m\}$ and $\mathrm{cd}(B) = \{1, q, r\}$.

3 Isaacs-Seitz's Conjecture

Let G be a group. Then the **derived subgroup** of G is

$$G' = \langle [a, b] = a^{-1}b^{-1}ab \mid a, b \in G \rangle.$$

For $m > 1$, the **m-th derived subgroup** of G is

$$G^{(m)} = (G^{(m-1)})'.$$

A group G is **solvable** if there exists a positive integer m such that $G^{(m)} = 1$ and the **derived length** of a solvable group G is the smallest positive integer m such that $G^{(m)} = 1$. In this case, we write $\mathrm{dl}(G) = m$.

Let χ be an irreducible $\mathbb{F}$–character of the finite group G. Then χ is **monomial** if $\chi = \lambda^G$ where λ is a linear $\mathbb{F}$–character of some subgroup of G. The group G is an **M-group (monomial group)** if every $\chi \in \mathrm{Irr}(G)$ is monomial.

Theorem 6. *(Taketa's Theorem) Let G be an M-group. Then G is solvable and*

$$\mathrm{dl}(G) \leq |\mathrm{cd}(G)|.$$

Isaacs-Seitz's Conjecture - Taketa's Inequality: *Let G be a solvable group. Then*

$$\mathrm{dl}(G) \leq |\mathrm{cd}(G)|.$$

Some known results about Isaacs-Seitz' Conjecture are:

- (M. Isaacs- D. Passman- 1969) If $|\mathrm{cd}(G)| = 3$, then G is solvable and $\mathrm{dl}(G) \leq 3$.

- (S. Garrison- 1973) If G is solvable and $|\mathrm{cd}(G)| = 4$, then $\mathrm{dl}(G) \leq 4$.
- (M. Isaacs- 1975) If G is solvable, then $\mathrm{dl}(G) \leq 3|\mathrm{cd}(G)|$.
- (Berger- 1976) If the group G has odd order, then Taketa's inequality holds.
- (D. Gluck-1985) If G is solvable, then $\mathrm{dl}(G) \leq 2|\mathrm{cd}(G)|$.
- (M. Lewis- 2001) If G is solvable and $|\mathrm{cd}(G)| = 5$, then $\mathrm{dl}(G) \leq 5$.
- (T. M. Keller- 2003) If G is solvable, then

  $$\mathrm{dl}(G) \leqslant |\mathrm{cd}(G)| + 24\log_2(|\mathrm{cd}(G)|) + 364.$$

- (K. Aziziheris- 2011) If $\mathrm{cd}(G) = \{1, a, b, c, ab, ac\}$, where a, b, and c are pairwise coprime integers greater than 1, then the derived length of G is at most 4.
- (T. Kildetoft- 2012) If G is solvable with $|\mathrm{cd}(G)| \geqslant 3$, then

  $$\mathrm{dl}(G) \leqslant 2|\mathrm{cd}(G)| - 3.$$

  One can see that Kildetoftresult is better when $|\mathrm{cd}(G)| \leqslant 587$ and Keller's result is better when $|\mathrm{cd}(G)| \leqslant 588$.
- (K. Aziziheris- M. Lewis- 2013) If all character degrees of G are odd, then G is solvable and $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$.
- (K. Aziziheris- M. Lewis- 2013) If all character degrees of G are even, then G is solvable and $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$.

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